

Some Remarks on Conway and Iteration Theories

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Abstract

We present an axiomatization of Conway theories which yields, as a corollary, a very concise axiomatization of iteration theories satisfying the functorial implication for base morphisms.

It has been shown that most fixed point operations in computer science share the same equational properties. These equational properties are captured by the notion of iteration theories [1, 3]. Several axiomatizations of iteration theories have been presented in [2, 3]. For a recent overview, we refer to [6].

The axioms of iteration theories can be conveniently divided into two groups: axioms for Conway theories and the commutative identities. The commutative identities have later been simplified to the group identities or certain generalized power identities, cf. [4, 5, 7]. In this note, we provide further axiomatizations of Conway and iteration theories, see Corollaries 3,4 and 5.

Iteration theories having a constructable fixed point operation (such as the theories of monotonic or continuous functions over a cpo or a complete lattice equipped with the least fixed point operation, or the theories of continuous functors over ω -categories equipped with the initial fixed point operation) usually satisfy the ‘functorial implication’ for base morphisms, see [2]. In fact, the commutative identities were introduced in [3] as a strictly weaker, but fully equational substitute for the functorial implication for (surjective) base morphisms, which are however still sufficient for completeness. As a corollary of our results, we obtain a simple axiomatization of iteration theories with a functorial dagger for base morphisms, cf. Corollary 6.

We assume familiarity with Conway and iteration theories and closely follow the terminology and notation in [2].

Proposition 1 *Let T be a preiteration theory. Then T is a Conway theory iff T satisfies the (base) parameter, fixed point, permutation and double dagger identities as well as the following identity:*

$$(\mathbf{1}_n \oplus 0_m) \cdot \langle f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p), g \rangle^\dagger = f^\dagger, \quad (1)$$

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where $f : n \rightarrow n + p$ and $g : m \rightarrow n + m + p$.

Proof. It is known that all identities mentioned in the proposition hold in Conway theories, see the summary on p. 212 and Proposition 3.18 on p. 134 in [2]. Suppose now that the identities mentioned in the proposition hold. We prove that the pairing identity

$$\langle f, g \rangle^\dagger = \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle, h^\dagger \rangle \quad (2)$$

holds for all $f : n \rightarrow n + m + p$ and $g : m \rightarrow n + m + p$, where

$$h = g \cdot \langle f^\dagger, \mathbf{1}_{m+p} \rangle : m \rightarrow m + p.$$

We follow closely the argument on p. 164 and 165 in [2]. The difference is that we use (1) instead of the ‘simplified composition identity’.

First, we establish the pairing identity in two special cases. The first special case is

$$\langle f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p), g \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p) \rangle^\dagger = \langle f^\dagger, g \cdot \langle f^\dagger, \mathbf{1}_p \rangle \rangle,$$

where $f : n \rightarrow n + p$ and $g : m \rightarrow n + p$. Indeed, let $h = \langle f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p), g \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p) \rangle$, then

$$(\mathbf{1}_n \oplus 0_m) \cdot h^\dagger = f^\dagger$$

by (1), so that $h^\dagger = \langle f^\dagger, k \rangle$ for some $k : m \rightarrow p$. Hence,

$$\begin{aligned} h^\dagger &= h \cdot \langle h^\dagger, \mathbf{1}_p \rangle \\ &= \langle f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p), g \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p) \rangle \cdot \langle f^\dagger, k, \mathbf{1}_p \rangle \\ &= \langle f \cdot \langle f^\dagger, \mathbf{1}_p \rangle, g \cdot \langle f^\dagger, \mathbf{1}_p \rangle \rangle \\ &= \langle f^\dagger, g \cdot \langle f^\dagger, \mathbf{1}_p \rangle \rangle \end{aligned}$$

by the fixed point identity.

The second special case is

$$\langle 0_n \oplus f, 0_n \oplus g \rangle^\dagger = \langle f \cdot \langle g^\dagger, \mathbf{1}_p \rangle, g^\dagger \rangle,$$

where $f : n \rightarrow m + p$ and $g : m \rightarrow n + p$. This follows from the first special case using the permutation identity (or the block transposition identity which is an instance of the permutation identity).

Now using the two special cases, the pairing identity (2) in full generality is established exactly as on p. 165 of [2].

Let $f : n \rightarrow n + m + p$ and $g : m \rightarrow n + m + p$, and define $h = g \cdot \langle f^\dagger, \mathbf{1}_{m+p} \rangle$ and $\tau = \langle \mathbf{1}_{n+m}, \mathbf{1}_{n+m} \rangle$. Then

$$\begin{aligned} \langle f, g \rangle^\dagger &= (\langle f \cdot (\mathbf{1}_n \oplus 0_{m+n} \oplus \mathbf{1}_{m+p}), g \cdot (\mathbf{1}_n \oplus 0_{m+n} \oplus \mathbf{1}_{m+p}) \rangle \cdot (\tau \oplus \mathbf{1}_p))^\dagger \\ &= \langle f \cdot (\mathbf{1}_n \oplus 0_{m+n} \oplus \mathbf{1}_{m+p}), g \cdot (\mathbf{1}_n \oplus 0_{m+n} \oplus \mathbf{1}_{m+p}) \rangle^{\dagger\dagger}, \end{aligned}$$

by the double dagger identity,

$$= \langle 0_n \oplus f^\dagger, g \cdot \langle 0_n \oplus f^\dagger, 0_n \oplus \mathbf{1}_{m+p} \rangle \rangle^\dagger,$$

by the first special case and the (base) parameter identity,

$$\begin{aligned} &= \langle 0_n \oplus f^\dagger, 0_n \oplus h \rangle^\dagger \\ &= \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle, h^\dagger \rangle, \end{aligned}$$

by the second special case. \square

Suppose that T is a preiteration theory and \mathcal{C} is a set of morphisms in T . Following [2], we say that T satisfies the functorial implication for \mathcal{C} if for all $f : n \rightarrow n + p$ and $g : m \rightarrow m + p$ in T and $\rho : n \rightarrow m$ in \mathcal{C} ,

$$f \cdot (\rho \oplus \mathbf{1}_p) = \rho \cdot g \Rightarrow f^\dagger = \rho \cdot g^\dagger.$$

Lemma 2 *Let T be a preiteration theory. Then the functorial implication holds in T for injective base morphisms iff the permutation identity and (1) hold.*

Proof. First note that permutation identity is equivalent to the special case of the functorial implication when ρ is a base permutation. If the permutation identity and (1) hold in T , then so does the functorial implication for injective base morphisms by the proof of Proposition 3.24 on p. 137 in [2]. Suppose now that the functorial implication holds for injective base morphisms. Then, as noted above, the permutation identity holds. To prove that (1) holds, suppose that $f : n \rightarrow n + p$ and $g : m \rightarrow n + m + p$, and let $h = \langle f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p), g \rangle : n + m \rightarrow n + m + p$. Then letting $\rho = \mathbf{1}_n \oplus 0_m$, we have

$$f \cdot (\rho \oplus \mathbf{1}_p) = \rho \cdot h,$$

so that

$$f^\dagger = \rho \cdot h^\dagger$$

by the functorial implication for injective base morphisms. \square

It is known that the permutation identity holds in a preiteration theory iff the block transposition identity does. This is due to the fact that every permutation $[n] \rightarrow [n]$ can be written as a composition of block transpositions, and if the permutation identity holds for bijective base morphisms $\pi_1, \pi_2 : n \rightarrow n$, then it also holds for $\pi_1 \cdot \pi_2 : n \rightarrow n$.

Corollary 3 *A preiteration theory T is a Conway theory iff the (base) parameter, fixed point, double dagger and permutation (or block transposition) identities and (1) hold in T .*

Proof. For one direction, recall that one of the axiomatizations of Conway theories consists of the left zero, right zero, pairing and permutation identities, cf. p. 212 in [2]. But the left zero identity is an instance of the fixed point identity, the right zero identity is an instance of the (base) parameter identity and the permutation identity follows from the functorial implication

for injective base morphisms. Moreover, the functorial implication for injective base morphisms follows from the block transposition identity and (1) as shown in Lemma 2.

For the other direction, recall that all properties mentioned in the corollary hold in Conway theories. \square

Corollary 4 *A preiteration theory T is a Conway theory iff the (base) parameter, fixed point, double dagger identities and the functorial implication for injective base morphisms hold in T .*

Corollary 5 *A preiteration theory T is an iteration theory iff it satisfies the fixed point, (base) parameter and double dagger identities and the functorial implication for injective base morphisms, moreover, it satisfies*

- *the commutative identities [3, 2] or*
- *the group identities [4], or*
- *the generalized power identities [5, 7].*

Corollary 6 *A preiteration theory T is an iteration theory satisfying the functorial implication for base morphisms iff the fixed point, (base) parameter and double dagger identities hold in T and T satisfies the functorial dagger implication for base morphisms.*

References

- [1] S.L. Bloom, C.C. Elgot, J.B. Wright: Solutions of the iteration equation and extensions of the scalar iteration operation. *SIAM J. Comput.* 9(1): 25-45 (1980)
- [2] S.L. Bloom, Z. Ésik: *Iteration Theories*. Springer, 1993
- [3] Z. Ésik: Identities in iterative and rational algebraic theories. *CL&CL*, 14: 183-207 (1980)
- [4] Z. Ésik: Group axioms for iteration. *Inf. Comput.* 148(2): 131-180 (1999)
- [5] Z. Ésik: Axiomatizing Iteration Categories. *Acta Cybern.* 14(1): 65-82 (1999)
- [6] Z. Ésik: Equational properties of fixed point operations in cartesian categories: an overview. In: *proc. MFCS (1)*, LNCS 9234, 18-37 (2015)
- [7] Z. Ésik: Equational axioms associated with finite automata for fixed point operations in cartesian categories. *MSCS*, published on line in 2015